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Endpoint Inequalities for Spherical Multilinear Convolutions

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Write $\sigma=(\sigma_1, ..., \sigma_n)$ for an element of the sphere Σ_{n-1} and let $d\sigma$ denote Lebesgue measure on Σ_{n-1}. For functions $f_1, ..., f_n$ on $\mathbf{R}$, define

Let $R=R(n)$ denote the closed convex hull in $\mathbf{R}^2$ of the points $(0, 0)$, $(1/n, 1)$, $((n+1)/(n+2), 1)$, $((n+1)/(n+3), 2/(n+3))$, $((n-1)/(n+1), 0)$. We show that if $n\geqslant 3$, then the inequality

$$\|T(f_1, ..., f_n)\|_q \leqslant C\, \|f_1\|_p \cdots \|f_n\|_p$$

holds if and only if $(1/p, 1/q)\in R$. Our results fill in the gap in the necessary and sufficient conditions when $n\geqslant 3$ in Oberlin's previous work.

A negative result is given along with some positive results, when $n=2$, thus narrowing the gap in the necessary and sufficient conditions in this case.

1. INTRODUCTION

Write $\sigma=(\sigma_1, ..., \sigma_n)$ for an element of the unit sphere Σ_{n-1} in $\mathbf{R}^n$, $n\geqslant 2$, and let $d\sigma$ denote Lebesgue measure on Σ_{n-1}. Define an n-linear operator T by

$$T(f_1, ..., f_n)(x)=\int_{\Sigma_{n-1}} f_1(x-\sigma_1)\cdots f_n(x-\sigma_n)\, d\sigma, \qquad x\in\mathbf{R},$$

for, say, bounded Borel functions $f_1, ..., f_n$ on $\mathbf{R}$.

We would like to consider the problem of determining all pairs $(1/p, 1/q)\in[0, 1]\times[0, 1]$ such that there is an inequality

$$\|T(f_1, ..., f_n)\|_q \leqslant C\, \|f_1\|_p \cdots \|f_n\|_p. \tag{1}$$

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Here $\|\cdot\|_p$ is the L^p norm with respect to Lebesgue measure on $\mathbf{R}$. Throughout, C denotes a positive constant which may not be the same at each occurrence, but which is always independent of the functions $f_1, ..., f_n$.

Let $R = R(n)$ denote the closed convex hull in $\mathbf{R}^2$ of the points $O = (0, 0)$, $F = (1/n, 1)$, $A = ((n+1)/(n+2), 1)$, $M = ((n+1)/(n+3), 2/(n+3))$, $B = ((n-1)/(n+1), 0)$. If $L = (n/(n+2), 0)$ and $W = (1, 1)$, then M is the point of intersection of the line segments AL and BW. (See Figure 1.) Oberlin [O1] proved the following result.

THEOREM A. *If* (1) *holds, then* $(1/p, 1/q)$ *lies in the region* R. *Conversely, if* $(1/p, 1/q)$ *is in the region* R *and not on the two closed line segments forming the right-hand boundary* D *of* R, *then* (1) *holds. If the functions* $f_1, ..., f_n$ *are restricted to be characteristic functions of subsets of* $\mathbf{R}$, *then* (1) *holds also when* $(1/p, 1/q) \in D$.

The purpose of this paper is to fill in the gap between the necessary and sufficient conditions in Theorem A when $n \geqslant 3$. That is, we have the following

THEOREM 1. *If* $n \geqslant 3$, *then* (1) *holds if and only if* $(1/p, 1/q) \in R$.

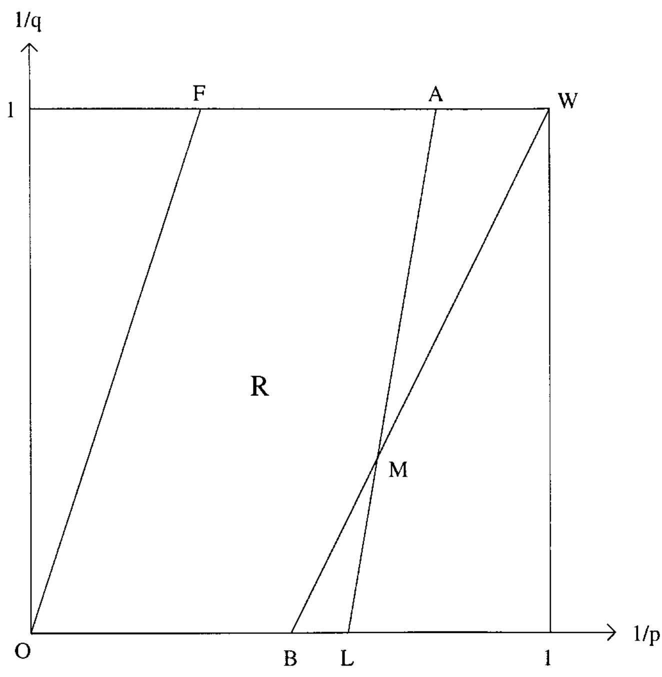

FIGURE 1

In view of the necessary condition and the positive results at O, F in Theorem A, Theorem 1 follows by the multilinear Riesz–Thorin theorem from parts (A), (B), (C) of the next result. Let us write $\|\cdot\|_{p,q}$ for the $L^{p,q}$ (Lorentz space) norm on $\mathbf{R}$. Recall that $L^{p,p} = L^p$ and $L^{p,q} \subset L^{p,r}$ if $q \leqslant r$. (See [SW, BS] for the basic definitions and facts on Lorentz spaces.)

PROPOSITION 2. (A) *If* $n \geqslant 2$, *then* (1) *holds when* $(1/p, 1/q) = A = ((n+1)/(n+2), 1)$.

(B) *If* $n \geqslant 3$, *then* (1) *holds when* $(1/p, 1/q) = B = ((n-1)/(n+1), 0)$.

(C) *If* $n \geqslant 3$, *then* (1) *holds when* $(1/p, 1/q) = M = ((n+1)/(n+3), 2/(n+3))$.

(D) *Let* $n = 2$ *and* $s > 0$. *There is an inequality*

$$\|T(f_1, f_2)\|_\infty \leqslant C\, \|f_1\|_{3,s}\, \|f_2\|_{3,s} \tag{2}$$

if and only if $s \leqslant 2$. *In particular,* (1) *fails if* $(1/p, 1/q) = B = (1/3, 0)$.

(E) *If* $n = 2$, *and* $H = (\frac{1}{2}, \frac{1}{4})$, $A = (\frac{3}{4}, 1)$, $M = (\frac{3}{5}, \frac{2}{5})$, *then* (1) *holds when* $(1/p, 1/q)$ *is on either of the closed line segments* AM *and* MH.

Part (D) contains a negative result in dimension 2, and part (E) gives some positive results in this case. (It is unknown at the moment whether (1) holds on the open segment BH when $n = 2$.)

The proof of part (A) is based on the methods in [O1] combined with the estimates in Lemma 2 and the so-called "multilinear trick" of Christ [C], which concerns multilinear interpolation in Lorentz spaces. (See Lemma 3 below. See also [O2].) A rough outline of the proof is as follows. We decompose the sphere Σ_{n-1} into certain subsets $G_1, ..., G_N$. (This decomposition is based on the fact that various favorable estimates, i.e., (2.1), (2.2) of Lemma 2 and (6), (7) below, are available if the domain of integration of T is restricted to suitable subsets.) Let T_j be the operator obtained by restricting the domain of integration of T to G_j. Writing χ_E for the characteristic function of a Borel set $E \subset \mathbf{R}$ and $|E|$ for the Lebesgue measure of E, we consider an estimate of the form

$$\|T_j(\chi_{E_1}, ..., \chi_{E_n})\|_1 \leqslant C\, |E_1|^{1/p_1} \cdots |E_n|^{1/p_n},$$

which is of course equivalent to the Lorentz space estimate

$$\|T_j(f_1, ..., f_n)\|_1 \leqslant C\, \|f_1\|_{p_1,1} \cdots \|f_n\|_{p_n,1}.$$

We establish this estimate when $(1/p_1, ..., 1/p_n) \in [0, 1]^n$ is any of the n vertices of a certain nontrivial $(n-1)$-simplex which contains the point $(1/p, ..., 1/p) = ((n+1)/(n+2), ..., (n+1)/(n+2))$ in the interior. This implies that T_j (and so T) is bounded from $L^p \times \cdots \times L^p$ to L^q when $(1/p, 1/q) = A = ((n+1)/(n+2), 1)$, by the multilinear trick mentioned above.

To carry out the proofs we need to consider an operator more general than T: given linearly independent vectors $v_1, ..., v_n$ in $\mathbf{R}^n$, define

$$T_v(f_1, ..., f_n)(x) = \int_{\Sigma_{n-1}} f_1(x - v_1 \cdot \sigma) \cdots f_n(x - v_n \cdot \sigma)\, d\sigma, \qquad x \in \mathbf{R},$$

where $v = (v_1, ..., v_n)$ and $x \cdot y$ stands for the dot product of $x, y \in \mathbf{R}^n$. Since our positive results are actually proved for this more general operator T_v, there is no danger of confusion in omitting the subscript in T_v in what follows.

Proposition 2(A) will be deduced from the next result, which contains the bulk of our work.

PROPOSITION 3. *Let $n \geqslant 2$. If $v_1, ..., v_n$ are linearly independent vectors in $\mathbf{R}^n$, then there exists a finite collection of sets $G_1, ..., G_N \subset \Sigma_{n-1}$ with $\Sigma_{n-1} = \bigcup_{j=1}^N G_j$ which satisfy the following properties*:

For each $j = 1, ..., N$, there exist points $Q_1, ..., Q_n$ in $[0, 1]^n$ such that their closed convex hull Σ is a nontrivial $(n-1)$-simplex which contains the point $Q(n) = ((n+1)/(n+2), ..., (n+1)/(n+2))$ in the interior and which is not parallel to any of the coordinate axes, and if

$$T_j(f_1, ..., f_n)(x) = \int_{G_j} f_1(x - v_1 \cdot \sigma) \cdots f_n(x - v_n \cdot \sigma)\, d\sigma, \tag{a}$$

then

$$\| T_j(\chi_{E_1}, ..., \chi_{E_n}) \|_1 \leqslant C \prod_{i=1}^n |E_i|^{1/p_i}, \tag{b}$$

whenever $(1/p_1, ..., 1/p_n)$ is in Σ.

In Section 2 we present three lemmas and then prove (parts (A), (B), and (D) of) Proposition 2 assuming Proposition 3. (The proofs of (C) and (E) of Proposition 2 are postponed until the end of the paper.) The proof of Proposition 3 is given in Section 3. We may clearly assume that all the functions f, g, and f_j appearing below are non-negative.

2. PROOF OF PROPOSITION 2

LEMMA 1. *Let $(Sg)(x) = \int_0^{2\pi} g(x - r \cos \theta)\, d\theta$, where $x, r \in \mathbf{R}$, and $r \neq 0$. Write $T(f, g)$ for $T(f_1, f_2)$. Then*

(1.1) $\|Sg\|_\infty \leqslant C\,\|g\|_{2,1}$,

(1.2) $\|T(f, g)\|_\infty \leqslant C\,\|f\|_{2,1}\,\|g\|_\infty$, $\|T(f, g)\|_\infty \leqslant C\,\|f\|_\infty\,\|g\|_{2,1}$,

(1.3) $\|T(f, g)\|_1 \leqslant C\,\|f\|_{2,1}\,\|g\|_1$, $\|T(f, g)\|_1 \leqslant C\,\|f\|_1\,\|g\|_{2,1}$.

Proof. (1.1) The change of variable $y = r\cos\theta$ and Hölder's inequality for Lorentz spaces give

$$\begin{aligned}(Sg)(x) &= 2\int_{-r}^{r} g(x-y)\,\frac{1}{\sqrt{r^2-y^2}}\,dy\\ &\leqslant C\,\|g\|_{2,1}\,\|(r^2-y^2)_+^{-1/2}\|_{2,\infty}\\ &\leqslant C\,\|g\|_{2,1}.\end{aligned}$$

(1.2) We have $T(f, g)(x) = \int_0^{2\pi} f(x - v_1\cdot e(\theta))\,g(x - v_2\cdot e(\theta))\,d\theta$, where $e(\theta) = (\cos\theta, \sin\theta)$. Write $v_j = r_j e(\phi_j)$ with $r_j = |v_j|$ for $j = 1, 2$. It follows from (1.1) that

$$\begin{aligned}T(f, g)(x) &= \int_0^{2\pi} f(x - r_1\cos(\theta-\phi_1))\,g(x - r_2\cos(\theta-\phi_2))\,d\theta\\ &\leqslant C\,\|f\|_{2,1}\,\|g\|_\infty.\end{aligned}$$

(1.3) Again by (1.1),

$$\begin{aligned}\|T(f, g)\|_1 &= \int_{-\infty}^{\infty} g(x)\int_0^{2\pi} f(x-(v_1-v_2)\cdot e(\theta))\,d\theta\,dx\\ &= \int_{-\infty}^{\infty} g(x)\int_0^{2\pi} f(x-|v_1-v_2|\cos(\theta-\mathrm{Arg}(v_1-v_2)))\,d\theta\,dx\\ &\leqslant C\,\|f\|_{2,1}\,\|g\|_1. \quad \blacksquare\end{aligned}$$

LEMMA 2. (2.1) *Let $n \geqslant 2$. For $\varepsilon > 0$ and $1 \leqslant j \leqslant n$, define*

$$U_j(f_1, ..., f_n)(x) = \int_{\{\sigma\in\Sigma_{n-1}\,:\,|v_j\cdot\sigma| > \varepsilon\}} \prod_{k=1}^{n} f_k(x - v_k\cdot\sigma)\,d\sigma.$$

Then

$$\|U_j(f_1, ..., f_n)\|_\infty \leqslant C\,\|f_j\|_\infty \prod_{k\neq j}\|f_k\|_1.$$

(2.2) *If $n \geqslant 3$, then*

$$\|T(f_1, ..., f_n)\|_\infty \leqslant C\,\|f_j\|_1 \prod_{k\neq j}\|f_k\|_{(n-1)/(n-3)}, \qquad 1 \leqslant j \leqslant n.$$

Proof. (2.1) It is enough to prove this for $j=n$. Rotate the axes such that v_1 points in the direction of $e_1=(1, 0, ..., 0)$. Rotate the axes again leaving e_1 fixed and choose $e_2=(0, 1, 0, ..., 0)$ such that v_2 lies in the linear span of $\{e_1, e_2\}$. Continuing in this way, we may parametrize the sphere Σ_{n-1} by $\sigma=\sum_{j=1}^n \sigma_j e_j$, where $\sigma_n=\pm\sqrt{1-\sigma_1^2-\cdots-\sigma_{n-1}^2}$, such that v_j lies in the linear span of $\{e_1, ..., e_j\}$, $1\leqslant j\leqslant n$. Thus

$$U_n(f_1, ..., f_n)(x)$$
$$=\int_{\{\sigma\in\Sigma_{n-1}:\,|v_n|\,|\sigma_n|>\varepsilon\}} \prod_{k=1}^n f_k\left(x-\sum_{j=1}^k c_{kj}\sigma_j\right)\frac{d\sigma_1\cdots d\sigma_{n-1}}{\sqrt{1-\sigma_1^2-\cdots-\sigma_{n-1}^2}}$$

for some constants $c_{kj}\in\mathbf{R}$. Here $c_{jj}\neq 0$, $1\leqslant j\leqslant n$, since $v_1, ..., v_n$ are linearly independent. So it follows that

$$\|U_n(f_1, ..., f_n)\|_\infty$$
$$\leqslant\frac{|v_n|}{\varepsilon}\|f_n\|_\infty\int_{[-1,1]^{n-1}}\prod_{k=1}^{n-1} f_k\left(x-\sum_{j=1}^k c_{kj}\sigma_j\right)d\sigma_1\cdots d\sigma_{n-1}$$
$$\leqslant C\,\|f_n\|_\infty\prod_{k=1}^{n-1}\|f_k\|_1,$$

where $C=\varepsilon^{-1}|v_n|\prod_{j=1}^{n-1}|c_{jj}|^{-1}$.

(2.2) We may assume that $j=1$. We will first show that

$$\|T(f_1, ..., f_n)\|_\infty\leqslant C\prod_{k=1}^n\|f_k\|_{p_k} \tag{3}$$

if two of the p_k are ∞ and the rest are 1. It is enough to do this for $(1/p_1, ..., 1/p_n)=(1, ..., 1, 0, 0)$. Using the coordinates introduced above and the polar coordinates in $\mathbf{R}^n$, we reparametrize Σ_{n-1} by

$$\sigma_1=\cos\phi_1$$
$$\sigma_2=\sin\phi_1\cos\phi_2$$
$$\cdots$$
$$\sigma_{n-1}=\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-2}\cos\phi_{n-1}$$
$$\sigma_n=\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-1},$$

where $0 \leqslant \phi_j \leqslant \pi$, $1 \leqslant j \leqslant n-2$, and $0 \leqslant \phi_{n-1} \leqslant 2\pi$. Then

$$d\sigma = (\sin \phi_1)^{n-2} (\sin \phi_2)^{n-3} \cdots (\sin \phi_{n-2})\, d\phi_1 \cdots d\phi_{n-1}.$$

Thus

$$T(f_1, ..., f_n)(x) \leqslant \|f_{n-1}\|_\infty \|f_n\|_\infty \int_0^{2\pi} \int_{[0,\pi]^{n-2}} \prod_{k=1}^{n-2} f_k\left(x - \sum_{j=1}^{k} c_{kj}\sigma_j\right)$$
$$\times (\sin \phi_1)^{n-2} (\sin \phi_2)^{n-3} \cdots (\sin \phi_{n-2})\, d\phi_1 \cdots d\phi_{n-2}\, d\phi_{n-1}$$

for the same constants c_{kj} as in the proof of (2.1). The change of variables $(\phi_1, ..., \phi_{n-2}) \mapsto (\sigma_1, ..., \sigma_{n-2})$ now shows that

$$T(f_1, ..., f_n)(x) \leqslant \|f_{n-1}\|_\infty \|f_n\|_\infty$$
$$\times \int_0^{2\pi} \int_{[-1,1]^{n-2}} \prod_{k=1}^{n-2} f_k\left(x - \sum_{j=1}^{k} c_{kj}\sigma_j\right) d\sigma_1 \cdots d\sigma_{n-2}\, d\phi_{n-1}$$
$$\leqslant C \|f_{n-1}\|_\infty \|f_n\|_\infty \prod_{k=1}^{n-2} \|f_k\|_1,$$

since the absolute value of the Jacobian is given by

$$\left|\frac{\partial(\sigma_1, ..., \sigma_{n-2})}{\partial(\phi_1, ..., \phi_{n-2})}\right| = (\sin \phi_1)^{n-2} (\sin \phi_2)^{n-3} \cdots (\sin \phi_{n-2}).$$

For $2 \leqslant i \leqslant n-1$, let $Q_i = (1, ..., 1, 0, 0, 1, ..., 1)$ be the vector in $\mathbf{R}^n$ with 0 in the ith and the $(i+1)$st places and 1 elsewhere. Also let $Q_n = (1, 0, 1, 1, ..., 1, 0)$ be the vector with 0 in the second and the nth places and 1 elsewhere. The above argument shows that (3) holds at Q_i, $2 \leqslant i \leqslant n$. Interpolating these estimates shows (2.2) with $j=1$, since $(1/(n-1))(Q_2 + \cdots + Q_n) = (1, (n-3)/(n-1), ..., (n-3)/(n-1))$. ∎

The following formulation of an observation of Christ (see pp. 227–228 in [C]) about multilinear interpolation of Lorentz spaces is due to Drury [D], where it is stated for $Y = \mathbf{C}$.

LEMMA 3. *Let Σ be a nontrivial closed $(n-1)$-simplex contained in $[0, 1]^n$. Assume that the hyperplane H generated by Σ is not parallel to any of the coordinates axes. Let Y be a Banach space and T an n-linear operator such that*

$$T: L^{p_1, 1} \times \cdots \times L^{p_n, 1} \to Y$$

is bounded whenever $(1/p_1, ..., 1/p_n)$ *is a vertex of* Σ. *Then for* $(1/p_1, ..., 1/p_n)$ *an interior point of* Σ *(relative to* H*) and* q_j *such that* $1 \leqslant q_j \leqslant \infty$ *and* $\sum_{j=1}^n (1/q_j) = 1$ *we have that*

$$T: L^{p_1, q_1} \times \cdots \times L^{p_n, q_n} \to Y$$

is a bounded map.

Proof of (A), (B), *and* (D) *of Proposition* 2. (A) Fix $n \geqslant 2$, and let G_j and T_j $(1 \leqslant j \leqslant N)$ be as in Proposition 3. (In fact, the case $n = 2$ follows from Hölder's inequality and the fractional integration theorem; see [O1].) Fix j. An application of Lemma 3 with $Y = L^1$ to the estimates (b) of Proposition 3 gives

$$\|T_j(f_1, ..., f_n)\|_1 \leqslant C \prod_{i=1}^n \|f_i\|_{(n+2)/(n+1), n} \leqslant C \prod_{i=1}^n \|f_i\|_{(n+2)/(n+1)}.$$

Since $\Sigma_{n-1} = \bigcup_{j=1}^N G_j$, we have $T(f_1, ..., f_n) \leqslant \sum_{j=1}^N T_j(f_1, ..., f_n)$. Therefore, summing the last estimates over j completes the proof of (A).

(B) Define U_j as in Lemma 2. Interpolating the estimates in (2.1) and (2.2) of Lemma 2, using the multilinear Riesz–Thorin theorem, yields

$$\|U_j(f_1, ..., f_n)\|_\infty \leqslant C \|f_1\|_{(n+1)/(n-1)} \cdots \|f_n\|_{(n+1)/(n-1)}$$

since $(2/(n+1))(0, 1) + ((n-1)/(n+1))(1, (n-3)/(n-1)) = ((n-1)/(n+1), (n-1)/(n+1))$. If $\varepsilon > 0$ is chosen small enough, then $T(f_1, ..., f_n) \leqslant \sum_{j=1}^n U_j(f_1, ..., f_n)$. So adding up the last estimates finishes the proof.

(D) Interpolating the estimate (2.1), for $n = 2$, $j = 1$, and (1.2) in Lemma 1, by using Lemma 3 with $Y = L^\infty$, yields

$$\|U_1(f, g)\|_\infty \leqslant C \|f\|_{3, 2} \|g\|_{3, 2}. \tag{4}$$

This, together with a similar estimate for U_2, implies that (2) holds if $s \leqslant 2$, because $\|f\|_{3, 2} \leqslant C \|f\|_{3, s}$ if $s \leqslant 2$.

To show that the condition $s \leqslant 2$ is necessary for (2) to hold, it suffices to show that (2) fails when $s \in (2, \infty)$. For simplicity we will do this for the original operator T. That is, we take $v_1 = e_1 = (1, 0)$, $v_2 = e_2 = (0, 1)$. Let us put

$$g(t) = \chi_{[0, 1/2]}(|t|) \, |t|^{-1/3} \, |\log |t||^{-1/2},$$

and $f(t)=g(t+1)$. Then

$$\|f\|_{3,s}^{s}=\|g\|_{3,s}^{s}=\int_0^\infty (t^{1/3}g^*(t))^s \frac{dt}{t} \approx \int_0^{1/2} |\log t|^{-s/2} \frac{dt}{t},$$

which converges for $s>2$. (Here g^* is the decreasing rearrangement of g.)
On the other hand, for some small positive constants ε and c,

$$T(f, g)(0) \geqslant c \int_0^\varepsilon y^{-1} |\log y|^{-1}\, dy=\infty.$$

Thus the monotone convergence theorem shows that $T(f, g)(x) \to \infty$ as $x \to 0^+$, and so (2) fails if $s>2$. ▮

3. PROOF OF PROPOSITION 3

We first present a lemma, which is used in proving Proposition 3 along with Lemmas 1 and 2 given in Section 2.

LEMMA 4. *Let $n \geqslant 3$, and $a \neq \pm 1$, $a \neq n-1$. Suppose that $M=\{b_{ij}\}$ is an $n \times n$ matrix such that the diagonal entries are* 0 *(i.e. $b_{ii}=0$), and for each $i=1, ..., n$, there exists some $j=1, ..., n$, $j \neq i$ such that $b_{ij}=1-a$. Suppose also that the remaining entries of M are* 1. *Then M is nonsingular.*

Proof. Let J be the $n \times n$ matrix with all the entries 1. Let us consider the matrix $M+xJ=\{b_{ij}+x\}$, for $x \in \mathbf{R}$. Note that $\sum_{j=1}^n b_{ij}=s=n-1-a \neq 0$, for $i=1, ..., n$. The determinant of $M+xJ$ does not change if the first $n-1$ columns are added to the nth column. This gives a new matrix M', all of whose nth column entries are $s+nx$. Therefore, we get

$$\det(M+xJ)=(s+nx)\, y,$$

where y is the determinant of the matrix that is obtained from M by replacing all the entries in the nth column by 1. We will now show that $y \neq 0$. If $M_1=-(M-J)=\{c_{ij}\}$, then M_1 has the diagonal entries $c_{jj}=1$, and each row contains exactly one entry of a, and all the remaining entries are 0. We may assume that no more than $n/2$ entries of a are above the diagonal. Eliminate the a in the first row by adding $-a$ times the first column to the column containing the a in the first row. Successive elimination of the nonzero entries above the diagonal, going down the rows, produces a lower triangular matrix, each of whose diagonal entries is either 1 or of the form $1-(-a)^m$, $2 \leqslant m \leqslant n$. Since $a \neq \pm 1$, we have $\det M_1 \neq 0$, and $y=(-1)^n (\det M_1)/(s-n)=(-1)^{n+1} (\det M_1)/(1+a) \neq 0$. Therefore, $\det M=sy=(n-1-a)\, y \neq 0$, proving Lemma 4. ▮

Proof of Proposition 3. If $n=2$, the proposition holds with $G_1=\Sigma_1$ and $Q_1=(1, \frac{1}{2})$, $Q_2=(\frac{1}{2}, 1)$. This is a consequence of (1.3) of Lemma 1.

Now suppose that the case $n-1$ of the proposition holds for some fixed $n \geqslant 3$. We will show that the proposition holds also for n. By mathematical induction this will imply that the proposition is true for all $n \geqslant 2$. (The reader may find it helpful to read the following proof first in dimension 3. In fact, in dimension 3 the proof can be simplified considerably.)

Following [O1] we define a linear map $\Lambda: \mathbf{R}^n \to \mathbf{R}^n$ by $\Lambda x=(v_1 \cdot x, ..., v_n \cdot x)$, $x \in \mathbf{R}^n$. Fix a unit vector u_n with $\Lambda u_n = c(1, ..., 1)$ for some $c \in \mathbf{R}$, and let $\{u_1, ..., u_n\}$ be an orthonormal basis for $\mathbf{R}^n$. Let $d\eta$ be Lebesgue measure on Σ_{n-2}. Parametrize Σ_{n-1} by

$$\sigma = \sum_{k=1}^{n-1} r\eta_k u_k + \operatorname{sgn}(r)\sqrt{1-r^2}\, u_n,$$

where $\eta=(\eta_1, ..., \eta_{n-1}) \in \Sigma_{n-2}$, $-1 \leqslant r \leqslant 1$. Then

$$d\sigma = d\eta\, |r|^{n-2} \frac{dr}{\sqrt{1-r^2}}$$

and

$$\begin{aligned} T(f_1, ..., f_n)(x) &= \int_{\Sigma_{n-1}} f_1(x-e_1 \cdot \Lambda\sigma) \cdots f_n(x-e_n \cdot \Lambda\sigma)\, d\sigma \\ &= \int_{-1}^{1} \int_{\Sigma_{n-2}} \prod_{j=1}^{n} f_j\Bigg(x-\operatorname{sgn}(r)\, c\sqrt{1-r^2} \\ &\qquad -re_j \cdot \Lambda\left(\sum_{k=1}^{n-1} \eta_k u_k\right)\Bigg)\, d\eta\, |r|^{n-2} \frac{dr}{\sqrt{1-r^2}}. \end{aligned}$$

Here $e_1, ..., e_n$ stand for the standard basis vectors in $\mathbf{R}^n$. Thus

$$\|T(f_1, ..., f_n)\|_1 = \int_{-1}^{1} F(r)\, |r|^{n-2} \frac{dr}{\sqrt{1-r^2}},$$

where

$$F(r) = \int_{-\infty}^{\infty} \int_{\Sigma_{n-2}} \prod_{j=1}^{n} f_j\left(x-re_j \cdot \Lambda\left(\sum_{k=1}^{n-1} \eta_k u_k\right)\right) d\eta\, dx.$$

The following inequality is (23) in [O1].

$$\int_{-1}^{1} F(r)\, |r|^{n-2}\, dr \leqslant C \prod_{k=1}^{n} \|f_k\|_1. \tag{5}$$

For some small $\varepsilon > 0$, let

$$T^1(f_1, ..., f_n)(x) = \int_{\{\sigma \in \Sigma_{n-1} : |\sigma \cdot u_n| > \varepsilon\}} f_1(x - v_1 \cdot \sigma) \cdots f_n(x - v_n \cdot \sigma)\, d\sigma$$
$$= \int_{-\sqrt{1-\varepsilon^2}}^{\sqrt{1-\varepsilon^2}} \int_{\Sigma_{n-2}} \prod_{j=1}^{n} f_j \left(x - \operatorname{sgn}(r)\, c \sqrt{1-r^2} \right.$$
$$\left. - r e_j \cdot \Lambda \left(\sum_{k=1}^{n-1} \eta_k u_k \right) \right) d\eta\, |r|^{n-2} \frac{dr}{\sqrt{1-r^2}},$$

and let $T^2 = T - T^1$. The following inequalities are (22) and (28) in [O1]:

$$\|T^1(f_1, ..., f_n)\|_1 \leqslant C \prod_{j=1}^{n} \|f_j\|_1, \tag{6}$$

$$\|T^2(\chi_{E_1}, ..., \chi_{E_n})\|_\infty \leqslant C \prod_{j=1}^{n} |E_j|^{n/(n+2)}. \tag{7}$$

In fact, (6) is an immediate consequence of (5), since

$$\|T^1(f_1, ..., f_n)\|_1 = \int_{-\sqrt{1-\varepsilon^2}}^{\sqrt{1-\varepsilon^2}} F(r)\, |r|^{n-2} \frac{dr}{\sqrt{1-r^2}}.$$

Note that (6) and (7) are estimates at the points W and L in Figure 1. Let us now give an outline of the inductive proof in [O1] that (1) holds at A when the f_j are characteristic functions. Assuming (1) holds at A in dimension $n-1$, the estimates (6) and (7), in dimension $n-1$, are interpolated with the estimates at B and A to show that (1) holds at M when the f_j are characteristic functions. The estimate at M in dimension $n-1$ is then used together with Hölder's inequality to prove estimates at A in dimension n. Our argument is similar, but in order to obtain n estimates for A, i.e., the estimates (b), we decompose T suitably and for each decomposed operator we replace the single estimates for T at B and A, used in the above outline, with the several estimates available at B (coming from Lemma 2) and A (coming from the induction hypothesis).

We have for $i = 1, ..., n$,

$$F(r) = r \int_{-\infty}^{\infty} f_i^r(x) \int_{\Sigma_{n-2}} \prod_{j \neq i} f_j^r(x - w_{ij} \cdot \eta)\, d\eta\, dx,$$

where $f_j^r(x) = f_j(rx)$ and

$$w_{ij} = ((e_j - e_i) \cdot \Lambda u_1, ..., (e_j - e_i) \cdot \Lambda u_{n-1}), \qquad 1 \leqslant j \leqslant n, \quad j \neq i.$$

It follows from the fact that $v_1, ..., v_n$ are linearly independent that, for each $i=1, ..., n$, the $n-1$ vectors w_{ij}, $1\leqslant j\leqslant n$, $j\neq i$, are linearly independent in $\mathbf{R}^{n-1}$. (See pp. 829–831 in [O1].)

Now set

$$S_i(f_1, ..., \tilde{f}_i, ..., f_n)(x)=\int_{\Sigma_{n-2}} \prod_{j\neq i} f_j(x-w_{ij}\cdot\eta)\, d\eta,$$

where the notation $\tilde{f}_i$ means that the f_i term is dropped. Observe that S_i is in the same form as T, but it is in dimension $n-1$. Thus we apply the above procedure of parametrizing Σ_{n-1} to Σ_{n-2}. Fix $i=1, ..., n$ and define a linear map $L_i\colon \mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$ by $L_i x=(w_1\cdot x, ..., (w_i\cdot x)^{\sim}, ..., w_n\cdot x)$, $x\in\mathbf{R}^{n-1}$, where the $w_i\cdot x$ term is dropped. Choose a unit vector $\mu_i\in\mathbf{R}^{n-1}$ so that $L_i\mu_i=c_i(1, ..., 1)$ for some $c_i\in\mathbf{R}$, and let $\{v_{i,1}, ..., v_{i,n-2}, \mu_i\}$ be an orthonormal basis for $\mathbf{R}^{n-1}$. Let $d\zeta$ be Lebesgue measure on Σ_{n-3}, and parametrize Σ_{n-2} by

$$\eta=\sum_{j=1}^{n-2} \rho\zeta_j v_{ij}+\operatorname{sgn}(\rho)\sqrt{1-\rho^2}\,\mu_i,$$

where $\zeta=(\zeta_1, ..., \zeta_{n-2})\in\Sigma_{n-3}$, $-1\leqslant\rho\leqslant 1$. (If $n=3$, take $\Sigma_{n-3}=\{-1, 1\}$.) Then

$$d\eta=d\zeta\;|\rho|^{n-3}\,\frac{d\rho}{\sqrt{1-\rho^2}}.$$

Now define subsets H_i of Σ_{n-2}, $1\leqslant i\leqslant n$, by

$$\begin{aligned} H_i&=\{\eta\in\Sigma_{n-2} : |\eta\cdot\mu_i|>\varepsilon\}\\ &=\Bigg\{\eta=\sum_{j=1}^{n-2}\rho\zeta_j v_{ij}+\operatorname{sgn}(\rho)\sqrt{1-\rho^2}\,\mu_i :\\ &\qquad\qquad (\zeta_1, ..., \zeta_{n-2})\in\Sigma_{n-3}, |\rho|<\sqrt{1-\varepsilon^2}\Bigg\}\end{aligned}$$

for some small $\varepsilon\in(0, 1)$, and let $K_i=\Sigma_{n-2}\backslash H_i$. Put

$$S_i^1(f_1, ..., \tilde{f}_i, ..., f_n)(x)=\int_{H_i}\prod_{j\neq i} f_j(x-w_{ij}\cdot\eta)\, d\eta,$$

and $S_i^2=S_i-S_i^1$, that is,

$$S_i^2(f_1, ..., \tilde{f}_i, ..., f_n)(x)=\int_{K_i}\prod_{j\neq i} f_j(x-w_{ij}\cdot\eta)\, d\eta.$$

It follows from the inequalities (6) and (7) in dimension $n-1$, respectively, that

$$\|S_i^1(f_1, ..., \tilde{f}_i, ..., f_n)\|_1 \leqslant C \prod_{j \neq i} \|f_j\|_1, \tag{8}$$

$$\|S_i^2(\chi_{E_1}, ..., \tilde{\chi}_{E_i}, ..., \chi_{E_n})\|_\infty \leqslant C \prod_{j \neq i} |E_j|^{(n-1)/(n+1)}. \tag{9}$$

We first decompose Σ_{n-2} into 2^n sets of the form $\bigcap_{i=1}^n B_i$, where each B_i may be chosen to be either K_i or H_i. Each B_i is decomposed further as follows. If B_i is K_i, let

$$B_{ij} = K_{ij} = K_i \cap G_{ij}, \qquad 1 \leqslant j \leqslant N,$$

where $G_{i1}, ..., G_{iN}$ are the sets coming from the induction hypothesis when the given vectors are $w_{i1}, ..., \tilde{w}_{ii}, ..., w_{in}$. Taking the union of K_{ij} over j gives K_i. If B_i is H_i, let

$$B_{ij} = H_{ij} = H_i \cap W_{ij},$$

where

$$W_{ij} = \{\eta \in \Sigma_{n-2} : |\eta \cdot w_{ij}| > \varepsilon\}, \qquad 1 \leqslant j \leqslant n, \quad j \neq i,$$

for a sufficiently small $\varepsilon > 0$. (Since for each fixed i, the $n-1$ vectors w_{ij}, $j \neq i$, are linearly independent in $\mathbf{R}^{n-1}$, it is easy to see that $\varepsilon > 0$ can be chosen so small that $\Sigma_{n-2} = \bigcup_{j \neq i} W_{ij}$. Hence, taking the union of H_{ij} over j gives H_i.)

Estimates for K_{ij}. Define

$$S_{ij}^2(f_1, ..., \tilde{f}_i, ..., f_n)(x) = \int_{K_{ij}} \prod_{k \neq i} f_k(x - w_{ik} \cdot \eta)\, d\eta. \tag{10}$$

The case $i = 1$ is typical. Fix j. Since $K_{1j} \subset G_{1j}$, it follows from the induction hypothesis that there exist points $Q_1, ..., Q_{n-1} \in [0, 1]^{n-1}$ and numbers $t_1, ..., t_{n-1} \in (0, 1)$, whose sum is 1, such that

$$\sum_{\ell=1}^{n-1} t_\ell Q_\ell = Q(n-1) = \left(\frac{n}{n+1}, ..., \frac{n}{n+1}\right), \tag{11}$$

and if we put $Q_\ell = (1/p_{\ell 2}, ..., 1/p_{\ell n})$, then

$$\|S_{1j}^2(\chi_{E_2}, ..., \chi_{E_n})\|_1 \leqslant C \prod_{k=2}^n |E_k|^{1/p_{\ell k}}, \qquad \ell = 1, ..., n-1. \tag{12}$$

Also, since $K_{1j} \subset K_1$, the inequality

$$\|S^2_{1j}(\chi_{E_2}, ..., \chi_{E_n})\|_\infty \leqslant C \prod_{k=2}^{n} |E_k|^{(n-1)/(n+1)} \tag{13}$$

follows from (9). By interpolation, (12) and (13) imply that

$$\|S^2_{1j}(\chi_{E_2}, ..., \chi_{E_n})\|_{(n+2)/2} \leqslant C \prod_{k=2}^{n} |E_k|^{\alpha_k}, \tag{14}$$

where

$$\alpha_k = \alpha_{\ell k} = \frac{n(n-1)}{(n+2)(n+1)} + \frac{2}{(n+2)\, p_{\ell k}}.$$

Write $F_*(r)$ for

$$\begin{aligned} F^2_{1j}(r) &= \int_{-\infty}^{\infty} \int_{K_{1j}} \prod_{k=1}^{n} f_k\left(x - re_k \cdot \Lambda\left(\sum_{m=1}^{n-1} \eta_m u_m\right)\right) d\eta\, dx \\ &= r \int_{-\infty}^{\infty} f^r_1(x)\, S^2_{1j}(f^r_2, ..., f^r_n)(x)\, dx. \end{aligned}$$

Put

$$\begin{aligned} T_*(f_1, ..., f_n)(x) &= T^2_{1j}(f_1, ..., f_n)(x) \\ &= \int_{-1}^{1} \int_{K_{1j}} \prod_{k=1}^{n} f_k\Bigg(x - \operatorname{sgn}(r)\, c\sqrt{1-r^2} \\ &\qquad - re_k \cdot \Lambda\left(\sum_{m=1}^{n-1} \eta_m u_m\right)\Bigg) d\eta\, |r|^{n-2} \frac{dr}{\sqrt{1-r^2}}. \end{aligned}$$

Then

$$\|T_*(f_1, ..., f_n)\|_1 = \int_{-1}^{1} F_*(r)\, |r|^{n-2} \frac{dr}{\sqrt{1-r^2}}. \tag{15}$$

Let us now set $f_k = \chi_{E_k}$ and write A_k for $|E_k|$. It follows from Hölder's inequality and (14) that

$$\begin{aligned} F_*(r) &\leqslant |r|\, \|f^r_1\|_{n+2/n}\, \|S^2_{1j}(f^r_2, ..., f^r_n)\|_{(n+2)/2} \\ &\leqslant C\, |r|\, |r|^{-n/(n+2)}\, A_1^{n/(n+2)} \prod_{k=2}^{n} (|r|^{-1} A_k)^{\alpha_k}. \end{aligned}$$

Hence,

$$F_*(r) \leqslant C\,|r|^{-b}\, A_1^{n/(n+2)} \prod_{k=2}^{n} A_k^{\alpha_k} \tag{16}$$

for some real number $b \leqslant n-2$ (for $n \geqslant 3$).

Now put $\delta = A_1^{2/(n+2)} \prod_{k=2}^{n} A_k^{1-\alpha_k}$. If $0 < \delta < 1/2$, we split the integral $\int_0^1 F_*(r)\, r^{n-2}(1-r^2)^{-1/2}\, dr$. It follows from (5) that

$$\begin{aligned}
\int_0^{1-\delta} F_*(r)\, r^{n-2} \frac{dr}{\sqrt{1-r^2}} &\leqslant C\delta^{-1/2} \int_0^1 F_*(r)\, r^{n-2}\, dr \\
&\leqslant C\delta^{-1/2} \prod_{k=1}^{n} A_k \\
&\leqslant C A_1^{(n+1)/(n+2)} \prod_{k=2}^{n} A_k^{\alpha + 1/((n+2)p_{\ell k})},
\end{aligned}$$

where

$$\alpha = \frac{1}{2} + \frac{n(n-1)}{2(n+2)(n+1)}.$$

Also, (16) implies that

$$\begin{aligned}
\int_{1-\delta}^{1} F_*(r)\, r^{n-2} \frac{dr}{\sqrt{1-r^2}} &\leqslant \int_{1-\delta}^{1} r^{n-2-b} \frac{dr}{\sqrt{1-r^2}}\, C A_1^{n/(n+2)} \prod_{k=2}^{n} A_k^{\alpha_k} \\
&\leqslant C\delta^{1/2} A_1^{n/(n+2)} \prod_{k=2}^{n} A_k^{\alpha_k} \\
&= C A_1^{(n+1)/(n+2)} \prod_{k=2}^{n} A_k^{\alpha + 1/((n+2)p_{\ell k})}.
\end{aligned}$$

On the other hand, if $\delta \geqslant 1/2$, then again by (16)

$$\begin{aligned}
\int_0^1 F_*(r)\, r^{n-2} \frac{dr}{\sqrt{1-r^2}} &\leqslant \int_0^1 r^{n-2-b} \frac{dr}{\sqrt{1-r^2}}\, C A_1^{n/(n+2)} \prod_{k=2}^{n} A_k^{\alpha_k} \\
&\leqslant C A_1^{n/(n+2)} \prod_{k=2}^{n} A_k^{\alpha_k} \\
&\leqslant C A_1^{(n+1)/(n+2)} \prod_{k=2}^{n} A_k^{\alpha + 1/((n+2)p_{\ell k})}.
\end{aligned}$$

Thus, in view of (15), we have just shown that

$$\|T_*(\chi_{E_1}, ..., \chi_{E_n})\|_1 \leqslant C \prod_{k=1}^{n} |E_k|^{\gamma_{\ell k}}, \tag{17}$$

where $T_* = T^2_{1j}$ and

$$(\gamma_{\ell 1}, ..., \gamma_{\ell n}) = P_\ell = \left(\frac{n+1}{n+2}, \alpha + \frac{1}{(n+2)\, p_{\ell 2}}, ..., \alpha + \frac{1}{(n+2)\, p_{\ell n}}\right),$$

for $\alpha = \frac{1}{2} + n(n-1)/(2(n+2)(n+1))$ and $\ell = 1, ..., n-1$. It follows from (11) that

$$\sum_{\ell=1}^{n-1} t_\ell P_\ell = Q(n) = \left(\frac{n+1}{n+2}, ..., \frac{n+1}{n+2}\right). \tag{18}$$

Estimates for H_{ij}. Define S^1_{ij} as in (10) with K_{ij} replaced by H_{ij}. Likewise define $F_* = F^1_{ij}$ and $T_* = T^1_{ij}$ as in the definitions right above (15), with obvious modifications. To simplify notation let us take, say, $i = 1$ and $j = 2$. By (8) we have

$$\|S^1_{1,2}(f_2, ..., f_n)\|_1 \leqslant C \prod_{k=2}^{n} \|f_k\|_1. \tag{19}$$

The following inequalities are immediate consequences of Lemma 2:

$$\|S^1_{1,2}(f_2, ..., f_n)\|_\infty \leqslant C \|f_2\|_\infty \prod_{k \neq 2} \|f_k\|_1 \qquad \text{if} \quad n \geqslant 3; \tag{20}$$

$$\|S^1_{1,2}(f_2, ..., f_n)\|_\infty \leqslant C \|f_2\|_1 \prod_{k \neq 2} \|f_k\|_{(n-2)/(n-4)} \qquad \text{if} \quad n \geqslant 4. \tag{21}$$

If $n = 3$, (21) should be replaced by

$$\|S^1_{1,2}(f_2, f_3)\|_\infty \leqslant C \|f_2\|_{2,1} \|f_3\|_\infty, \tag{21$'$}$$

which follows from (1.2) of Lemma 1.

If $n \geqslant 3$, then (19) and (20) imply that

$$\|S^1_{1,2}(f_2, ..., f_n)\|_{(n+2)/2} \leqslant C \|f_2\|_{(n+2)/2} \prod_{k=3}^{n} \|f_k\|_1.$$

Therefore, arguing exactly as one deduces (17) from (14), we obtain

$$\int_0^1 F_*(r)\, r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leqslant C\, |E_1|^{(n+1)/(n+2)}\, |E_2|^{(n+4)/(2(n+2))} \prod_{k=3}^{n} |E_k|,$$

where we again set $f_k = \chi_{E_k}$. Thus we have shown that $T_* = T^1_{1,2}$ satisfies (17) at

$$(\gamma_{0,1}, ..., \gamma_{0,n}) = P_0 = \left(\frac{n+1}{n+2}, \frac{n+4}{2(n+2)}, 1, ..., 1\right).$$

Similarly, if $n \geqslant 4$, (19) and (21) imply that $T^1_{1,2}$ satisfies (17) at

$$(\gamma_{1,1}, ..., \gamma_{1,n}) = P_1 = \left(\frac{n+1}{n+2}, 1, \beta, ..., \beta\right),$$

where

$$\beta = \frac{1}{2} + \frac{n^2 - 2n - 4}{2(n+2)(n-2)}.$$

Note that

$$\frac{2}{n} P_0 + \frac{n-2}{n} P_1 = Q(n) = \left(\frac{n+1}{n+2}, ..., \frac{n+1}{n+2}\right).$$

When $n = 3$, $i = 1$, and $j = 2$, we get $P_0 = (\frac{4}{5}, \frac{7}{10}, 1)$, and $P_1 = (\frac{4}{5}, \frac{17}{20}, \frac{7}{10})$, which satisfy

$$\tfrac{1}{3} P_0 + \tfrac{2}{3} P_1 = Q(3) = (\tfrac{4}{5}, \tfrac{4}{5}, \tfrac{4}{5}).$$

If $T_* = T^1_{ij}$, $j \neq i$, an analogous argument shows that, when $n \geqslant 4$, T_* satisfies (17) at the points P^i_0, P^i_1, where P^i_0 has $(n+1)/(n+2)$ in the ith place, $(n+4)/(2(n+2))$ in the jth place, and 1 elsewhere, and P^i_1 has $(n+1)/(n+2)$ in the ith place, 1 in the jth place, and β elsewhere.

Completion of the Proof of Proposition 3. Consider the finite collection of sets

$$\Big\{\sigma = \sum_{i=1}^{n-1} r\eta_i u_i + \operatorname{sgn}(r)\sqrt{1-r^2}\, u_n \in \Sigma_{n-1} :$$
$$\eta = (\eta_1, ..., \eta_{n-1}) \in B_{1, j_1} \cap \cdots \cap B_{n, j_n}, \ -1 \leqslant r \leqslant 1\Big\},$$

where each set B_{ij} (writing j for j_i) may be either K_{ij} or H_{ij}, and the indices $j_1, ..., j_n$ range over all possible values. Call these sets $G_1, ..., G_N$. Clearly, we have $\Sigma_{n-1} = \bigcup_{k=1}^{N} G_k$. Let $T_1, ..., T_N$ be the associated operators given by (a).

Fix $k = 1, ..., N$. Let $B_{1, j_1} \cap \cdots \cap B_{n, j_n}$ be the set associated with T_k. First suppose that there is some $i = 1, ..., n$ such that the set B_{ij} is K_{ij}. We may assume $i = 1$ without loss of generality. Let $P_1, ..., P_{n-1}$ be the points obtained above for K_{1j}, at which (17) holds for $T_* = T^2_{1j}$. Since $T_k \leqslant T^2_{1j}$, it follows that T_k satisfies (b) at these points. Each of the points $P_1, ..., P_{n-1}$ has the first coordinate $x_1 = (n+1)/(n+2)$, and their convex hull is a non-trivial $(n-2)$-simplex, which contains the point $Q(n)$ in the interior by (18).

We can pick two additional points P', P'', at which T_k satisfies (b), as follows. If $B_{mj} = H_{mj}$ for some $m = 2, ..., n$, then let P', P'' be the points that were obtained above for H_{mj}, at which (17) holds for $T_* = T^1_{mj}$. And if $B_{mj} = K_{mj}$ for all $m = 2, ..., n$, then let Σ' be the $(n-2)$-simplex with vertices at $P'_1, ..., P'_{n-1}$ (with $x_n = (n+1)/(n+2)$) at which (17) holds for T^2_{nj}. Pick two points P', P'' from Σ' such that P', P'' satisfy the conditions that $(1-t)\, P' + tP'' = Q(n)$ for some $0 < t < 1$ and that P', P'' have $x_1 \neq (n+1)/(n+2)$. This last condition can be satisfied, for it follows from the induction hypothesis that Σ' is not parallel to any of the coordinate axes.

Now let Ω be the closed convex hull of the selected points $P_1, ..., P_{n-1}$, P', P''. Then T_k satisfies (b) in all of Ω, by interpolation (or by taking suitable asymmetric geometric means of (b) at $P_1, ..., P_{n-1}$, P', and P''). Thus it follows that in this case Ω contains a nontrivial $(n-1)$-simplex Σ satisfying the required properties.

Suppose next that B_{ij} is H_{ij} for all $i = 1, ..., n$. Then the statement at the end of the section "Estimates for H_{ij}" implies that T_k satisfies (b) at the n pairs of points P^i_0, P^i_1, $1 \leqslant i \leqslant n$. When $n \geqslant 4$, the point P^i_0 has $(n+1)/(n+2)$ in the ith place, $(n+4)/(2(n+2))$ in the jth place for some $j \neq i$, and 1 elsewhere, and P^i_1 has $(n+1)/(n+2)$ in the ith place, 1 in the jth place for the same j as above, and β elsewhere. So the line segment $P^i_0 P^i_1$ is parallel to the vector $V_i = (b_{i1}, ..., b_{in})$, where $b_{ii} = 0$ and $b_{ij} = 1 - n/2$ for some $j \neq i$ and the remaining $n-2$ components are 1. An application of Lemma 4 with $a = n/2$ and $M = \{b_{ij}\}$ shows that $V_1, ..., V_n$ are linearly independent. So the linear span of the points P^i_0, P^i_1, $1 \leqslant i \leqslant n$, has dimension n. Thus the closed convex hull Ω of the points P^i_0, P^i_1, $1 \leqslant i \leqslant n$, contains an open ball centered at $Q(n)$, and so Ω certainly contains a non-trivial $(n-1)$-simplex Σ satisfying the requirements. (When $n = 3$, the points P^i_1 take a slightly different form. For example, when $i = 1$ and $j = 2$, we have $P^1_0 = (\frac{4}{5}, \frac{7}{10}, 1)$, and $P^1_1 = (\frac{4}{5}, \frac{17}{20}, \frac{7}{10})$. But V_i still have the same form, e.g. $V_1 = (0, -1/2, 1)$. So we may apply Lemma 4 with $a = 3/2$.)

Therefore, we may conclude that the proposition holds for n, as desired. This completes the proof of Proposition 3. ∎

Remark. Given $n \geqslant 3$, the above proof shows that T_k satisfies (b) in an open ball in $\mathbf{R}^n$ centered at $Q(n)$ for most values of k. However, there is also some T_k, say T_1, such that (b) holds only when $(1/p_1, ..., 1/p_n)$ lies on the hyperplane $x_1 + \cdots + x_n = n(n+1)/(n+2)$. This is the case when all the B_{ij} are K_{ij} in the sets $\bigcap_{i=1}^m B_{i,j_i}(m)$ $(m = 3, ..., n)$ which appear in the $n-2$ induction steps leading up to G_1 and T_1. Thus the above proof is consistent with the necessary condition in Theorem A. ▮

Now we will give the proof of Proposition 2(C). (In fact, the proof of this result is implicitly contained in the proof of Proposition 3.)

Proof of (C) *and* (E) *of Proposition* 2. (C) Write $T = T^1 + T^2$ as in the part preceding (6). It follows from (6) and Proposition 2(B) that T^1 is bounded at M, that is, bounded from $L^p \times \cdots \times L^p$ to L^q when $(1/p, 1/q) = M$. So it is enough to show that T^2 is bounded at M. With G_j and T_j as in Proposition 3, let

$$T_j^2(f_1, ..., f_n)(x) = \int_{G_j \cap \{\sigma \in \Sigma_{n-1} : |\sigma \cdot u_n| < \varepsilon\}} f_1(x - v_1 \cdot \sigma) \cdots f_n(x - v_n \cdot \sigma)\, d\sigma,$$

where u_n is the unit vector defined right above (5). Interpolating the estimates (b) at Q_ℓ in Proposition 3 and (7) gives

$$\|T_j^2(\chi_{E_1}, ..., \chi_{E_n})\|_{(n+3)/2} \leqslant C\, |E_1|^{1/p_{\ell 1}} \cdots |E_n|^{1/p_{\ell n}},$$

where

$$\left(\frac{1}{p_{\ell 1}}, ..., \frac{1}{p_{\ell n}}\right) = P_\ell = \frac{2}{n+3}\, Q_\ell + \frac{n+1}{n+3}\left(\frac{n}{n+2}, ..., \frac{n}{n+2}\right), \qquad 1 \leqslant \ell \leqslant n.$$

From the fact that $\sum_{\ell=1}^n t_\ell Q_\ell = Q(n) = ((n+1)/(n+2), ..., (n+1)/(n+2))$, for some numbers $t_\ell \in (0, 1)$, whose sum is 1, it follows that $\sum_{\ell=1}^n t_\ell P_\ell = ((n+1)/(n+3), ..., (n+1)/(n+3))$. Thus the convex hull of $P_1, ..., P_n$ is a nontrivial $(n-1)$-simplex containing the point $((n+1)/(n+3), ..., (n+1)/(n+3))$ in the interior. An application of Lemma 3 with $Y = L^{(n+3)/2}$ now shows that T_j^2 is bounded at M. Therefore, we conclude that T^2 is bounded at M.

(E) The proof that (1) holds on AM and MH is similar to the proof of (C). We omit the details. ▮

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